

Characterization of curves in $C^{(2)}$

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Abstract

In this paper we characterize the irreducible curves lying in $C^{(2)}$. We prove that a curve B has a degree one morphism to $C^{(2)}$ with image a curve of degree d with irreducible preimage in $C \times C$ if and only if there exists an irreducible smooth curve D and morphisms from D to C and B of degrees d and 2 respectively forming a diagram which does not reduce.

Keywords: Symmetric product, curve, irregular surface, curves in surfaces.

1 Introduction

Given a curve $B \subset C^{(2)}$, we define the **degree** of B as the integer d such that $C_P \cdot B = d$ where C_P denotes the coordinate curve in $C^{(2)}$ with base point P . The curves of degree one in $C^{(2)}$ are completely characterized by the two results in [ACGH85, Pg. 310, D-10] and [Cil83], where it is proven that a curve B of degree one in $C^{(2)}$ different from a coordinate curve is smooth and it exists if and only if there exists a degree two morphism $f : C \rightarrow B$. Moreover, $B = \{f^{-1}(q) \mid q \in B\} \subset C^{(2)}$.

In [Cha08] a different proof of this result is given. From this proof we remark that considering the curve $B \subset C^{(2)}$ as before, then, the preimage of B by $\pi_C : C \times C \rightarrow C^{(2)}$ is isomorphic to C through the projection onto the first factor.

Let \tilde{B} be an irreducible curve in $C^{(2)}$ different from a coordinate divisor. Let B be its normalization and assume that there is no degree two morphism from C to B . Then, since C_P is ample in $C^{(2)}$, from the characterization of degree one curves we deduce that $\tilde{B} \cdot C_P \geq 2$. In this paper we present a characterization of curves with any degree. First of all we need the following definition:

Definition 1.1. *We say that a diagram of morphisms of curves*

$$\begin{array}{ccc} D & \xrightarrow{(e:1)} & B \\ (d:1) \downarrow & & \\ C & & \end{array}$$

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reduces if there exist curves F and H such that there exists a diagram

$$\begin{array}{ccc}
 D & \xrightarrow{(e:1)} & B \\
 \downarrow (k:1) & & \downarrow (k:1) \\
 F & \xrightarrow{(e:1)} & H \\
 \downarrow (d:1) & & \\
 C & &
 \end{array}$$

with $k > 1$, the upper square being a commutative diagram and the left vertical arrows giving a factorization of the original degree d morphism.

When $k = d$ we will say that the diagram **completes**, and we will obtain a commutative diagram

$$\begin{array}{ccc}
 D & \xrightarrow{(e:1)} & B \\
 \downarrow (d:1) & & \downarrow (d:1) \\
 C & \xrightarrow{(e:1)} & H.
 \end{array}$$

Notice that when d is a prime number both definitions are equivalent.

In this paper we prove

Theorem 1.2. *Let B be an irreducible smooth curve such that there are no non-trivial morphisms $B \rightarrow C$. A morphism of degree one from the curve B to the surface $C^{(2)}$ exists, with image \tilde{B} of degree d if, and only if, there exists a smooth irreducible curve D and a diagram*

$$\begin{array}{ccc}
 D & \xrightarrow{(2:1)} & B \\
 \downarrow (d:1) & & \\
 C & &
 \end{array}$$

which does not reduce.

If we consider the case $d = 1$ we recover the results for degree one.

We prove the theorem in two steps, giving a separated proof for each implication (see Theorem 2.2 and Theorem 2.3). First, given a diagram

$$\begin{array}{ccc}
 D & \xrightarrow[f]{(2:1)} & B \\
 \downarrow (d:1) g & & \\
 C & &
 \end{array}$$

which does not reduce we find a curve in $C^{(2)}$ defined by it as the image by $g^{(2)}$ of the immersion of B in $D^{(2)}$ given by f . We prove that $g^{(2)}|_B$ has degree one, and

hence the curve in $C^{(2)}$ has normalization B and the normalization map is precisely $g^{(2)}|_B$. Second, given a curve lying in $C^{(2)}$ we find a diagram defined by the curve, its preimage by π_C and the projection on one factor of $C \times C$. We compute the degrees of the different maps and prove that this diagram does not reduce.

In a following paper we are going to use this result to study and classify curves of degree two and some of degree three.

Notation: We work over the complex numbers. By curve we mean a complex projective reduced algebraic curve. Let C be a smooth curve of genus $g \geq 2$, we put $C^{(2)}$ for its 2nd symmetric product. We denote by $\pi_C : C \times C \rightarrow C^{(2)}$ the natural map, and $C_P \subset C^{(2)}$ a coordinate curve with base point $P \in C$. We put Δ_C for the main diagonal in $C^{(2)}$, and $\Delta_{C \times C}$ denotes the diagonal of the Cartesian product $C \times C$.

2 Characterization

We begin with a lemma that will simplify the rest of the exposition.

Lemma 2.1. *We consider a diagram of morphisms of smooth irreducible curves*

$$\begin{array}{ccc} D & \xrightarrow[(2:1)]{f} & B \\ (d:1) \downarrow g & & \\ C & & \end{array} .$$

The image of $B \subset D^{(2)}$ (with the immersion given by the fibers of f) by the morphism $g^{(2)}$ is the diagonal $\Delta_C \subset C^{(2)}$ if and only if the morphism g factorizes through the curve B by f .

Proof. Let i be the involution on D that defines f , that is, the change of sheet. Since $B = \{x + y \mid f(x) = f(y)\} = \{x + i(x)\} \subset D^{(2)}$, then $\text{Im}(g^{(2)}|_B) = \{g(x) + g(i(x))\}$. It is contained in the diagonal Δ_C if and only if $g(x) = g(i(x))$ for all $x \in D$, that is, if and only if g factorizes through B by f . \square

In the following theorem, given a diagram that does not reduce we deduce the existence of a curve in $C^{(2)}$ naturally attached to it.

Theorem 2.2. *Assume that there exists a diagram of morphisms of smooth irreducible curves*

$$\begin{array}{ccc} D & \xrightarrow[(2:1)]{f} & B \\ (d:1) \downarrow g & & \\ C & & \end{array}$$

which does not reduce and such that the morphism g does not factorize through B by f . Then, $g^{(2)}$ gives a degree one map $B \rightarrow C^{(2)}$ with reduced image a curve \tilde{B} of degree precisely d .

Proof. Consider a diagram as above and look at the induced morphism $D^{(2)} \xrightarrow{g^{(2)}} C^{(2)}$. As we have seen in the Introduction, we have an immersion $B \subset D^{(2)}$ as the set of pairs of points in D with the same image by f . Then, we consider D inside $D \times D$ as $\pi_D^{-1}(B) \cong D$, that is, ordered pairs of points with the same image by f .

Let $\tilde{B} = g^{(2)}(B)_{red}$, the reduced image curve in $C^{(2)}$, and consider the map $B \xrightarrow{(k:1)} \tilde{B}$ induced by $g^{(2)}$. We want to see that $k = 1$.

Notice that by Lemma 2.1 we can assume that \tilde{B} is not Δ_C . We know that $B \cdot D_P = 1$, hence,

$$1 = g^{(2)*}(B \cdot D_P) = g^{(2)*}(B) \cdot \left(\frac{1}{d}C_P\right) \Rightarrow g^{(2)*}(B) \cdot C_P = d.$$

In addition, since the map $B \xrightarrow{(k:1)} \tilde{B}$ is $g^{(2)}|_B$, we obtain that $d = (k\tilde{B}) \cdot C_P$, and thus $\tilde{B} \cdot C_P = \frac{d}{k}$, that is, k divides d .

Assume by contradiction that $k > 1$.

Let F be the preimage of \tilde{B} by the morphism $\pi_C : C \times C \rightarrow C^{(2)}$. Then $F \rightarrow \tilde{B}$ has degree two and thus we obtain a diagram

$$\begin{array}{ccc} D \times D & \xrightarrow{\pi_D} & D^{(2)} \\ \uparrow \wr & & \uparrow \wr \\ D & \xrightarrow[f]{(2:1)} & B \\ \downarrow & & \downarrow (k:1) \\ F & \xrightarrow{(2:1)} & \tilde{B} \\ \downarrow & & \downarrow \\ C \times C & \xrightarrow{\pi_C} & C^{(2)} \end{array} \quad \begin{array}{l} (1) \\ \\ \\ \\ \\ \end{array}$$

Observe that the exterior arrows form a commutative diagram, and hence, also the interior arrows give a commutative diagram. Thus, the morphism $D \rightarrow F$ has degree k . Now, the restriction to D of $g \times g$ followed by the projection onto one factor of $C \times C$ is precisely $g : D \rightarrow C$ by construction. That is, we obtain the diagram

$$\begin{array}{ccc} D & \xrightarrow{(2:1)} & B \\ \downarrow (k:1) & & \downarrow (k:1) \\ F & \xrightarrow{(2:1)} & \tilde{B} \\ \downarrow g & & \\ C & & \end{array} .$$

Hence, the original diagram reduces, contradicting our hypothesis.

Consequently, $k = 1$ and thus we deduce that the curve \tilde{B} has normalization B .

Moreover, looking at diagram (1) we deduce that $D \xrightarrow{(1:1)} F$, that is, the preimage of \tilde{B} by π_C has normalization D , and we will denote it by \tilde{D} . So we have:

$$\begin{array}{ccc}
 D \times D & \xrightarrow{\pi_D} & D^{(2)} \\
 \uparrow \scriptstyle g \times g & \swarrow \scriptstyle f & \uparrow \scriptstyle g^{(2)} \\
 \tilde{D} & \dashrightarrow & B \\
 \downarrow \scriptstyle g \downarrow & & \downarrow \scriptstyle g^{(2)} \\
 \tilde{D} & \longrightarrow & \tilde{B} \\
 \downarrow \scriptstyle g \downarrow & & \downarrow \scriptstyle g^{(2)} \\
 C \times C & \xrightarrow{\pi_C} & C^{(2)} \\
 \downarrow \scriptstyle pr & & \\
 C & &
 \end{array} \tag{2}$$

where the dashed arrows show the original diagram. \square

Conversely, we have also a theorem in the opposite direction, from the existence of curves in $C^{(2)}$ we deduce the existence of diagrams which do not reduce.

Theorem 2.3. *Given an irreducible curve \tilde{B} lying in $C^{(2)}$ with degree d , let B be its normalization, and assume that there are no non trivial morphisms $B \rightarrow C$. Then, there exists a smooth irreducible curve D and a diagram*

$$\begin{array}{ccc}
 D & \xrightarrow{(2:1)} & B \\
 (d:1) \downarrow & & \\
 C & &
 \end{array}$$

which does not reduce.

Proof. First of all, we observe that \tilde{B} is not the diagonal in $C^{(2)}$ because we are assuming that there are no morphisms from B to C .

Let $\tilde{D} = \pi_C^*(\tilde{B}) \in \text{Div}(C \times C)$ and D its normalization. We notice that with our hypothesis \tilde{D} is irreducible. Indeed, otherwise, one of its components would have as normalization the curve B , because we have a $(2 : 1)$ morphism from \tilde{D} to B , and since $\tilde{D} \subset C \times C$ we would obtain a non trivial morphism from B to C contradicting our hypothesis.

Now, we are going to compute the degree of $\tilde{D} \rightarrow C$, given by the projection onto one factor:

$$\begin{aligned}
 \tilde{D} \cdot (C \times P + P \times C) &= \pi_{C*}(\pi_C^*(\tilde{B}) \cdot \pi_C^*(C_P)) = \\
 &= \tilde{B} \cdot \pi_{C*}\pi_C^*(C_P) = 2\tilde{B} \cdot C_P = 2d.
 \end{aligned}$$

And therefore, since \tilde{D} is symmetric with respect to the involution $(x, y) \rightarrow (y, x)$ by construction, $\tilde{D} \cdot (C \times P) = d$, and so, the degree of the morphism on C is precisely d . In this way, we have a diagram

$$\begin{array}{ccc} \tilde{D} \xrightarrow{(2:1)} \tilde{B} & \text{and taking their normalizations} & D \xrightarrow{(2:1)} B \\ (d:1) \downarrow & \text{we obtain a diagram of} & (d:1) \downarrow \\ C & \text{morphisms of smooth curves} & C \end{array} .$$

We call $f : D \rightarrow B$ the map coming from $\pi_C|_{\tilde{D}}$ and $g : D \rightarrow C$ the map coming from the projection onto one factor of $C \times C$.

Let α be the degree one morphism induced in B by the immersion of \tilde{B} in $C^{(2)}$. Since we have $D \xrightarrow{(2:1)} B$, as we have seen in the Introduction there exists an immersion of B in $D^{(2)}$ as pairs of points with the same image by this morphism.

Since $D \xrightarrow{(1:1)} \tilde{D} \subset C \times C$ we can consider that a general point in D is a pair (x, y) with $x, y \in C$. Moreover, since $D \rightarrow B$ is induced by $\pi_C|_{\tilde{D}}$, a general fiber of $D \rightarrow B$ will be two points (x, y) and (y, x) . Hence, we can write a general point of $B \subset D^{(2)}$ as $(x, y) + (y, x)$.

Now, we consider the restriction to $B \subset D^{(2)}$ of $g^{(2)}$. By construction this morphism is precisely α and therefore the image is the original \tilde{B} . In particular, $g^{(2)}|_B$ is generically of degree one.

We are going to see that the diagram does not reduce by contradiction: Assume that there exist curves F and H , and a diagram

$$\begin{array}{ccc} D & \xrightarrow{f} & B \\ \downarrow h & & \downarrow r \\ F & \xrightarrow{s} & H \\ \downarrow l & & \\ C & & \end{array}$$

as in Definition 1.1. Then, as we have seen in the Introduction, the fibers of s give a curve isomorphic to H inside $F^{(2)}$. Hence, we have

$$\begin{array}{ccccc} & & & \tilde{B} & \\ & & \alpha & \nearrow & \\ B & \hookrightarrow & D^{(2)} & \xrightarrow{g^{(2)}} & C^{(2)} \\ \downarrow & & \downarrow h^{(2)} & \nearrow l^{(2)} & \\ H & \hookrightarrow & F^{(2)} & & \end{array}$$

By definition, the image of $B \subset D^{(2)}$ by $h^{(2)}$ is $H \subset F^{(2)}$, that is, the embedding of H in $F^{(2)}$ given by s , and we know that $l \circ h = g$ so $l^{(2)} \circ h^{(2)} = g^{(2)}$, hence

$$g^{(2)}|_B : B \xrightarrow{h^{(2)}=r} H \xrightarrow{l^{(2)}} \tilde{B}$$

(1:1)

thus r , as well as h , have degree one. Consequently, our diagram does not reduce (see Definition 1.1). \square

We observe that we could change the hypothesis of the non existence of morphisms from B to C by assuming that \tilde{B} is not the diagonal and that $\pi_C^{-1}(\tilde{B})$ is irreducible. Putting these two theorems together we find the characterization of curves in the symmetric square $C^{(2)}$ previously stated in Theorem 1.2.

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